The Cardiff Conundrum

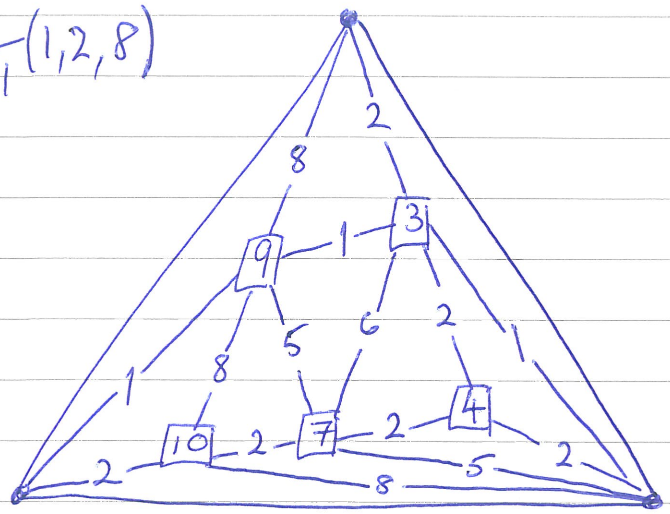
(jt with Jesus Tapia Amador)

Plan

1. Ancient history (= May 2000)
2. The Conundrum
3. Combinatorial Reid's recipe
4. Localisation algorithm.

§1 Ancient history

$\frac{1}{11}(1,2,8)$



G-Hilb for

finite abelian $G \subset SL(3,\mathbb{C})$

Smooth toric var.

$G\text{-Hilb} \longrightarrow \mathbb{C}^3/G.$

Reid's recipe: marks internal nodes & edges with irred. reps of G. ($\rho \neq \rho_0$).
(local geom of surface determines ρ on nodes)

Fact: each $\rho \neq \rho_0$ appears once

Geometry: $\exists$ tautological line bundles $\{L_\rho : \rho \in \mathrm{Irr}(G)\}$

$$\mathrm{Pic}(G\text{-Hilb}) = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z} \cdot L_\rho \Big/ \left\{ \begin{array}{l} L_4 \simeq L_2 \otimes L_2 \\ L_3 \simeq L_1 \otimes L_2 \\ L_{10} \simeq L_2 \otimes L_8 \end{array} \right\}$$

Restriction:

$$\{ L_\rho|_{\mathbb{P}^2} : \rho \in \mathrm{Irr}(G) \} /_{\sim \mathrm{isom.}} = \{ \mathcal{O}_{\mathbb{P}^2}(i) : 0 \leq i \leq 2 \}$$

Observation [C-King, Logvinenko].

$\forall$ compact $S_n \subset Y = G\text{-Hilb}$, bundle

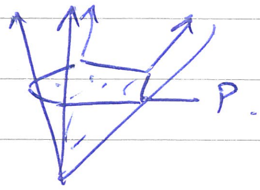
$$T = \overline{\bigoplus_{\rho \in \mathrm{Irr}(G)} L_\rho}\Big|_{S_n} \quad \text{tilting on } S_n$$

i.e. $R\mathrm{Hom}(T, -) : D^b(\mathrm{coh}(S_n)) \longrightarrow D^b(\text{mod-}\mathrm{End}(T))$

Question: Why?

§2 The Conundrum

Generalise from lattice triangle to lattice ^convex^ polygon.



__Theorem__ [Ishii–Ueda]

Let P be a lattice polygon.

$\exists$ (consistent) quiver Q, $A := \Bbbk Q/(\text{ideal})$

① $Z(A)$ semigp alg $\Bbbk[\mathrm{Cone}(P) \cap \mathbb{Z}^3]$

② Fix vertex $0 \in Q_0$; $\exists$ distinguished crep. res

$$Y = M_\theta(\text{mod-}A) \longrightarrow X = \operatorname{Spec} Z(A)$$

det. by triangulation Σ of P.

③ $\exists$ taut. line bundles $\{L_i : i \in Q_0\}$ s.t.

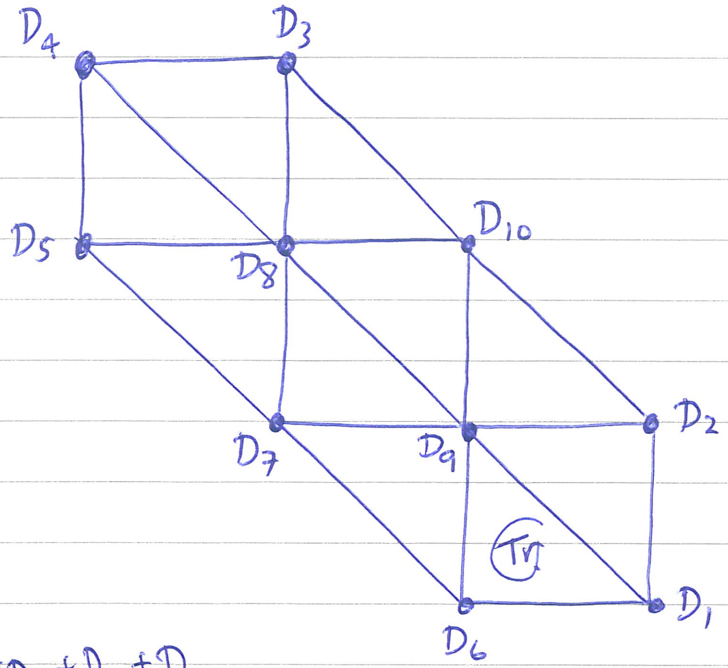
(i) $\bigoplus_{i \in Q_0} L_i$ is tilting on Y

(ii) $A \cong \operatorname{End}(\bigoplus_{i \in Q_0} L_i)$

$\quad\;\; \cong$

$\quad \Bbbk Q/(\text{ideal})$

arrow a $\qquad L_{t(a)} \xrightarrow{\cdot D_a} L_{h(a)}$


D_4
D_3
D_5
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Tr
D_6
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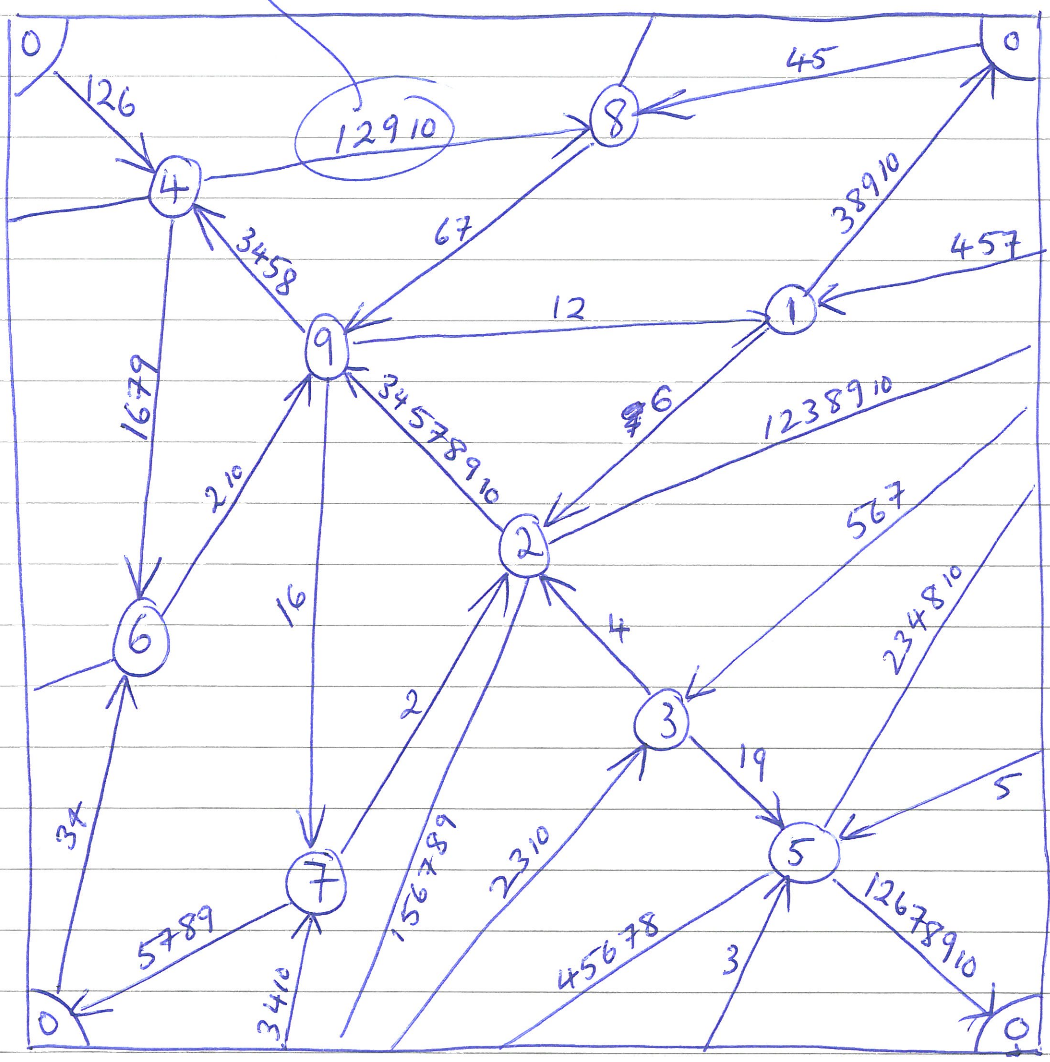
$D_1+D_2+D_9+D_{10}$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7
156789
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5789
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The Conundrum

EXCEPT ONE

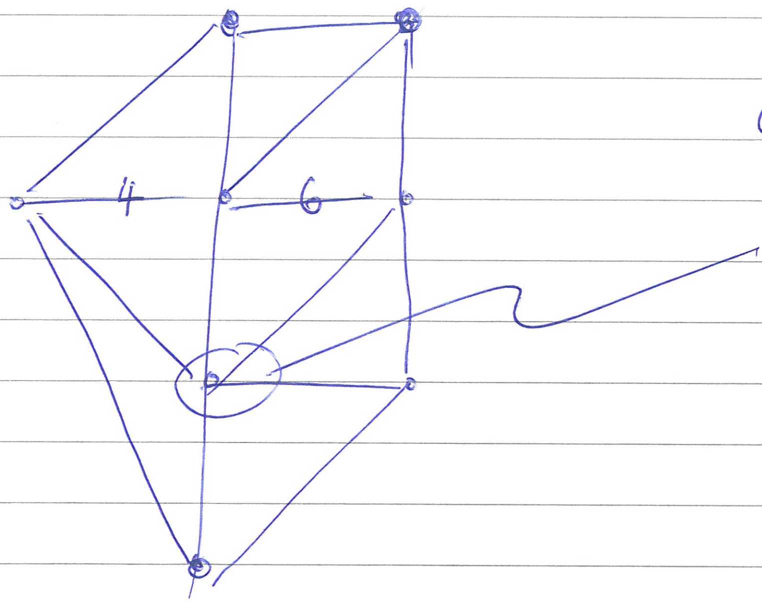
For all known compact surfaces $S_n \subset Y\big|_{X}$,

the bundle

$$T := \overline{\bigoplus_{i \in Q_0}} L_i \Big|_{S_n}$$

is a tilting bundle on S_n.

Y

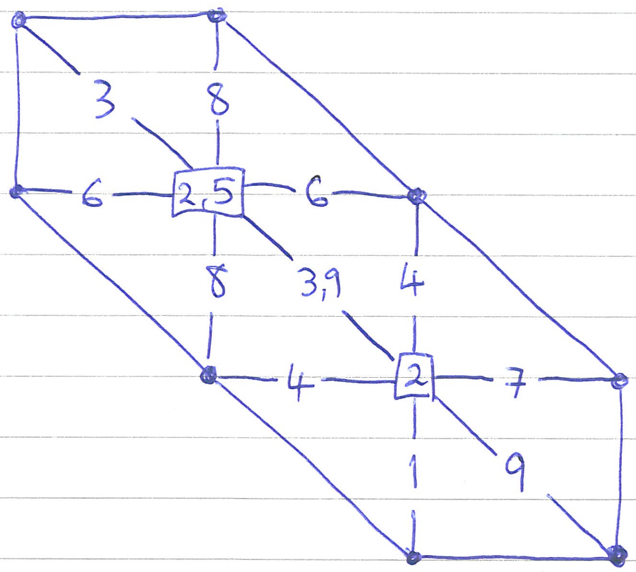


counterexample

!!

§3 Combinatorial Reid's recipe.

Running example:



Notes.

① $i \in Q_0$ labelling edge not unique;

② $i \in Q_0$ labels node, may label other nodes

③ not determined by local geometry.

Proposition.

Q (consistent) dimer model quiver, choose $0 \in Q_0$. Can mark nodes & edges (interior) with vertices of Q in a manner generalising RR for G-Hilb.

Rmk: Not obvious a priori that each $i \in Q_0$ appears.

§4 Localisation Algorithm

Choose $\alpha \in Q_1$ with $\text{tail}(\alpha) = 0$.

$D_\alpha \coloneqq$ divisor labelling α.

Lemma: D_α is sum of divisors for nodes in sector of boundary of P.

Def: $\Sigma' \subset \Sigma$: remove triangles touching these nodes

Notice:

(1) $h(\alpha)$ labels all edges in $\partial\Sigma' \cap \Sigma^\circ$.

(2) map $L_0 \xrightarrow{\cdot D_\alpha} L_{h(\alpha)}$ nowhere zero on Y'

$$L_0|_{Y'} \cong L_{h(\alpha)}|_{Y'}.$$

$\therefore$ on Y' must identify $0 = t(\alpha)$ with $h(\alpha)$.

$\therefore$ $\#\{\text{vertices of } Q\}$ drops by 1 if that's all we do

$$S_\alpha \coloneqq \{a \in Q_1 \mid D_\alpha - D_a \geq 0\}$$

$$L_{t(a)}|_{Y'} \cong L_{h(a)}|_{Y'}.$$

<u>Facts:</u>

① $a \in S_\alpha \setminus \{\alpha\}$ has $\begin{cases} t(a) \text{ labelling edge in } \Sigma \setminus \Sigma' \\ h(p) \text{ labelling node in } \partial\Sigma' \cap \Sigma^\circ . \end{cases}$

p starts with a

② $\forall$ node $n \in \partial \Sigma'$

$$\sum_{\substack{t(a) \text{ labels} \\ \text{edge crossing} \\ \text{at node } n}} D_p = D_\alpha$$

ie. $\bigotimes_{\cancel{t(a)}} L_{h(p)} \otimes L_{t(p)}^{-1} = L_{h(\alpha)} \otimes \cancel{L_0}^{-1}$

$$\boxed{\text{ie} \quad \bigotimes_{\substack{i \text{ labels} \\ \text{edges thro' } n}} L_i = \bigotimes_{\substack{j \text{ label} \\ \text{nodes}}} L_j}$$

Key Arrow Contraction Algorithm:

Thm

(Assume each internal node of Σ has valency ≤ 7)

For $S_\alpha := \{ a \in Q_1 \mid D_\alpha - D_a \geqslant 0 \}$ and

$Q' = \overline{Q[S_\alpha^{-1}]}$

the algebra $A' = \Bbbk Q'/(\text{ideal})$ is a

(consistent) dimer model algebra.

$\Longrightarrow \bigoplus_{i \in Q_0'} L_i'$ tilting a $Y' = Y - \mathrm{Supp}(D_\alpha)$.

Algorithm:

① Every $i \in Q_0$ appears once a Σ.

② Given compact $S_n \subset Y$ weak Fano, if convex polygon is end-point of sequence of contractions, $\bigoplus_{i \in Q_0} L_i|_{S_n}$ is tilting.